Nome, Cognome, Matricola:

1. Sia $\alpha \in \mathbb{R}$ e sia $f : \mathbb{R} \to \mathbb{R}$ data da

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{\arctan(x)} & \text{se } x < 0 \\ 0 & \text{se } x = 0 \\ \dfrac{\sin(x^2) - \log(1+x^2)}{x^{\alpha+4}} & \text{se } x > 0 \end{cases}$$

Allora f è continua in $x = 0$ se

[A] : $\alpha \leq 2$ [B] : $\alpha \leq 1$ [C] : $\alpha \leq 0$ [D] : $\alpha < 1$ [X] : $\alpha < 0$

Punteggio: 7

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2. Il limite

$$\lim_{n\to\infty} \frac{n^2\left(1 - n^{\frac{3}{n}}\right)}{\log(1+n^n)}$$

vale

[A] : 1 [B] : -2 [C] : -1 [X] : -3 [E] : 2

Punteggio: 7

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3. Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che

$$|z| = 5 \quad \text{e} \quad z \cdot \frac{\exp\left(i\frac{\pi}{2}\right) - 1}{|z|^4 + \exp(|z|^2) + 1} \in (-\infty, 0)$$

Punteggio: 6

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4. La serie numerica

$$\sum_{n=1}^{\infty} n^{\alpha} \Big(\log(1+n^3) - 3\log(n) \Big)^2$$

converge se e solo se

[A] : $\alpha < 5$ [X] : $\alpha \leq 4$ [C] : $\alpha < 4$ [D] : $\alpha \leq 5$ [E] : $\alpha < 6$

Punteggio: 6

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5. Calcolare l'integrale

$$\int_0^1 \frac{e^{4x}}{e^{8x} + 1} \, \mathrm{d}x$$

Punteggio: 6

Svolgimento

1.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\arctan(x) - x}{x \arctan(x)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{3}x^3}{x^2} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2 - \frac{x^6}{6} - x^2 + \frac{x^4}{2}}{x^{\alpha+4}}$$

$$= \lim_{x\to 0^+} \frac{\frac{x^4}{2}}{x^{\alpha+4}} = 0 \iff \alpha < 0$$

2.

$$1 - n^{\frac{3}{n}} = 1 - e^{\frac{3}{n}\log n} \quad \left(\frac{3}{n}\log n \to 0\right)$$

$$= 1 - \left(1 + \frac{3}{n}\log n + o\left(\frac{1}{n}\log n\right)\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2\left(1 - n^{\frac{3}{n}}\right)}{\log(1+n^n)} = \lim_{n\to\infty} \frac{n^2\left(-\frac{3}{n}\log n + o(n\log n)\right)}{\log(n^n)}$$

$$= \lim_{n\to\infty} \frac{-3n\log(n)}{n\log(n)} = -3$$

3. $z = x + iy \qquad |z| = 5 \iff x^2 + y^2 = 25$

$$e^{i\frac{\pi}{2}} - 1 = i - 1$$

$$z \cdot (e^{i\frac{\pi}{2}} - 1) = (x+iy)(i-1) = -x - y + i(x-y)$$

$$< 0$$

$$\Rightarrow \quad x = y \quad \wedge \quad x > 0$$

$$\Rightarrow \quad 1 \text{ ponto} : \quad \boxed{z = \frac{5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}}$$

4.

$$\log(1+n^3) - 3\log n = \log(1+n^{-3}) \sim n^{-3} \qquad n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} n^{\alpha} \left(\log(1+n^3) - 3\log(n)\right)^2$$

$$\sim \sum_{n=1}^{\infty} n^{\alpha-6} \qquad \text{converge} \iff \boxed{\alpha < 5}$$

5.

$$t = e^{4x}$$

$$dt = 4e^{4x}\,dx$$

$$\int_0^1 \frac{e^{4x}}{1+e^{8x}}\,dx = \frac{1}{4}\int_1^{e^4} \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{4}\left(\operatorname{arctan}(e^4) - \frac{\pi}{4}\right)$$